Lecture 24

In this lecture we complete our study of logic, by examining what happens to the truth values of various predicate formulae when the domain of interpretation changes. As we shall see, a change in the meaning of the predicate may also affect the truth values.

As before, we consider the predicate $P(x,y)$ defined to mean $x \geqslant y$.

We now give the name D_1 to our first domain of interpretation $\mathbb{N} \setminus \{0\} = \{1, 2, 3, 4, \ldots\}$.

Here are six formulae of predicate logic.

(i) $\forall x \forall y \ P(x,y)$

(ii) $\forall x \exists y \ P(x,y)$

(iii) $\exists x \forall y \ P(x,y)$

(iv) $\exists x \exists y \ P(x,y)$

(v) $\forall y \exists x \ P(x,y)$

(vi) $\exists y \forall x \ P(x,y)$

We've seen that with D_1 as domain, the following truth values apply:

(i) F

(ii) T

(iii) F

(iv) T

(v) T

(vi) T

But what happens if we change to another domain?

Let D_2 be the finite set $\{1, 2, 3, 4, 5\}$, and let D_3 be the set of all integers (so that $D_3 = \mathbb{Z}$).

Now what can we say about the truth values of the six predicate formulae?

Changing from D_1 to D_2 only causes one of the truth values to change — the truth value for formula (iii) becomes true. With $P(x,y)$ as currently defined, what this formula says is that "the domain has a largest element x". Clearly, when the domain is D_1 this statement is false but it becomes true whenever the domain is a finite set of real numbers such as D_2.

Can the statement be true for an infinite set? Yes! For example, the set of all negative integers is infinite and has a largest element -1.

Now let's suppose that the domain is D_3. Here are the truth values:

(i) F
(ii) T
(iii) F
(iv) T
(v) T
(vi) F

Notice that formula (vi) has now become false. What this formula says is that "the domain has a smallest element y". Both D_1 and D_2 have a smallest element, but D_3 doesn't. That's why the formula is now false.

Is there a domain D_4 for which all six formulae are true?

Yes. We need D_4 to be a singleton, such as $D_4 = \{1\}$. With this domain the predicate $x \geqslant y$ always means $1 \geqslant 1$, which is true. So formula (i) is true, and as a consequence the other five formulae are also true.

Now suppose that we keep our original domain D_1, but change the meaning of $P(x,y)$ to "$x > y$". What happens to the truth values of the six predicate formulae?

Here are the new truth values:

(i) F

(ii) F (Given $x=1$, there is no y such that $x>y$.)

(iii) F

(iv) T

(v) T

(vi) F (Although $y=1$ is the smallest number, it's not less than <u>all</u> elements of D_1 because it's not less than <u>itself</u>.)

If we keep this new $P(x,y)$ but change the domain, can we make (iv) false?

Yes — if $D_4 = \{1\}$ is the domain then (iv) is false, and as a consequence the other five formulae are also false.